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The role of boundary conditions in solving finite-energy, two-body, bound-state Bethe–Salpeter equations ☆

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Abstract

The difficulties that typically prevent numerical solutions from being obtained to finite-energy, two-body, bound-state Bethe–Salpeter equations can often be overcome by expanding solutions in terms of basis functions that obey the boundary conditions. The method discussed here for solving the Bethe–Salpeter equation requires only that the equation can be Wick rotated and that the two angular variables associated with rotations in three-dimensional space can be separated, properties that are possessed by many Bethe–Salpeter equations including all two-body, bound-state Bethe–Salpeter equations in the ladder approximation. The efficacy of the method is demonstrated by calculating finite-energy solutions to the partially-separated Bethe–Salpeter equation describing the Wick–Cutkosky model when the constituents do not have equal masses.

Keywords: Bethe–Salpeter equation; Relativistic equations; Bound-state equations

1. Introduction

The Bethe–Salpeter equation [1] is a covariant equation that, in some sense, is a relativistic generalization of the Schrödinger equation although it is developed from relativistic quantum field theory rather than from relativistic quantum mechanics. One particularly noteworthy feature of the equation is that interactions are retarded so that there is no action at a distance. While the Bethe–Salpeter equation is appropriate for studying properties of relativistic bound-state systems, heretofore its use has been limited because, even numerically, the two-body, bound-state equation has been exceedingly difficult to solve [2]. For this reason various approximations such as the Blankenbecler–Sugar approximation [3] or the instantaneous approximation [1,4] are often made that reduce the covariant equation in four-dimensional space–time to a more tractable, approximately-covariant equation in three dimensions.

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If there are no external fields, the Bethe–Salpeter equation is rotationally invariant in three-dimensional space so two angular variables can be separated. Furthermore, at least in the ladder approximation, the equation can be Wick rotated (analytically continued to Euclidean space) [5], which eliminates the singularity in the kernel and makes the equation much easier to solve. When a Bethe–Salpeter equation has been partially separated and Wick-rotated, it is still an integral or differential equation in two variables. The numerical method discussed here offers the possibility of obtaining finite-energy solutions even when the equation is not completely separable, which is usually the case, and does not require that the masses of the two bound quanta be equal.

With several exceptions, solutions to the two-body, bound-state Bethe–Salpeter equation have been obtained in the ladder approximation only when the equation is completely separable or when the masses of the two bound quanta are equal. The Wick–Cutkosky model, which consists of two unequal-mass scalars interacting via a massless scalar, is completely separable [5–10], and the eigenvalue equation for the coupling constant can be solved numerically [6,11,12]. In the zero-energy limit the Bethe–Salpeter equation is rotationally invariant in four-dimensional space–time and is therefore separable. Sometimes the completely separated equation has been solved numerically. For example, Brennan [13] obtained zero-energy, bound-state solutions for two unequal-mass fermions interacting via a massive scalar. In the ladder approximation the author [14,15] calculated zero-energy solutions for a spin-0 and spin-1/2 constituent with masses that are not equal and are bound by scalar electrodynamics.

Even if the equation is not completely separable, finite-energy, two-body, bound-state solutions can occasionally be obtained if the masses of the two bound quanta are equal. For example, Gammel and Menzel [16] determined the bound-state solutions of two oppositely charged fermions that interact through minimal electrodynamics. Schwartz [17] and Nieuwenhuis and Tjon [18] determined bound-state solutions for two scalars that interact via a third, massive scalar. When the two bound scalars have unequal masses, finite-energy solutions were first obtained by Kaufmann [19] and later by Setô and Fukui [20], who reduce the Bethe–Salpeter equation to an infinite system of integral equations in one variable that are solved numerically. In all cases, before the equations are solved, they are Wick-rotated [5] to eliminate the singularity in the kernel.

Because the energy appears more than once in the Bethe–Salpeter equation, a Hamiltonian does not exist, and the equation is an eigenvalue equation for the coupling constant instead of the energy. The equation is solved by specifying a value for the energy and then, for the chosen value of energy, calculating values of coupling constant that satisfy the equation. Although the coupling constants are real in the Lagrangian, there are apparently solutions to the Bethe–Salpeter equation with complex values of the coupling constant. While the Wick–Cutkosky model [5,6] has only real values for the coupling constant, Kaufmann [19] considered two scalars interacting with a third, massive scalar and found complex values. Also, for the same equation Setô and Fukui [20] found that "there exists a strong indication that complex eigenvalues appear..." Here attention is restricted to solutions of the Bethe–Salpeter equation with real values of the coupling constants, which are the more interesting physically.

Numerical solutions to the bound-state, Bethe–Salpeter equation are obtained in five steps:

(1) The singularity in the kernel is removed by a Wick rotation [5], which is always possible in the ladder approximation, and is accomplished by making the substitution $p_0 \to \mathrm{i}p_0$ while rotating the path of integration 90° counterclockwise in the complex p_0-plane.

(2) Two angular variables are separated, which is possible because the Bethe–Salpeter equation is rotationally invariant in three-dimensional space provided there are no external fields. The resulting equation for the Bethe–Salpeter "wave function" $\Psi(\mathrm{i}p_0, p_s)$ is an equation in the two variables p_0 and $p_s \equiv |\mathbf{p}|$. In the ladder approximation a Wick-rotated, partially-separated Bethe–Salpeter equation is of the form

$$K(\mathrm{i}p_0, p_s)\Psi(\mathrm{i}p_0, p_s) = \frac{g_1 g_2}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}q_0 \int_0^{\infty} \mathrm{d}q_s V(\mathrm{i}p_0, p_s, \mathrm{i}q_0, q_s)\Psi(\mathrm{i}q_0, q_s). \tag{1.1}$$

The above equation actually represents N_{EQ} equations, where $N_{\mathrm{EQ}} = 1$ if both constituent quanta have spin zero and $N_{\mathrm{EQ}} > 1$ otherwise. Thus, $K(\mathrm{i}p_0, p_s)$ and the kernel $V(\mathrm{i}p_0, p_s, \mathrm{i}q_0, q_s)$ are both $N_{\mathrm{EQ}} \times N_{\mathrm{EQ}}$ matrix functions.

(3) Zero-energy solutions are calculated. In the zero-energy limit the Bethe–Salpeter equation is invariant under rotations in four-dimensional space–time, and is, therefore, completely separable. Zero-energy solutions are expanded in terms of basis functions that consist of the product of a set of basis functions $\{g_i(|p|)\}$ that depend on the magnitude of the Euclidean four-momentum $|p| = (p_0^2 + p_s^2)^{1/2}$ and hyperspherical harmonics in four-dimensional, Euclidean space–time. To obtain solutions, each of the basis functions $g_i(|p|)$ must (very nearly) obey the boundary conditions, which are readily calculated [15]. Each basis function need not obey the boundary conditions exactly provided that a linear combination of the basis functions yields a solution that does.

(4) Finite-energy solutions $\Psi(\mathrm{i}p_0, p_s)$ are expanded in terms of a set of basis functions $\{g_j(p_0, p_s)\}$,

$$\Psi(\mathrm{i}p_0, p_s) = \sum_{j=1}^{N_{\mathrm{B}}} c_j g_j(p_0, p_s). \tag{1.2}$$

Two conditions are imposed on the basis functions: (a) The basis functions must (very nearly) obey the boundary conditions. (b) A basis system must be chosen that, in the zero-energy limit, devolves to the basis system that yields zero-energy solutions. Knowledge of a basis system that yields zero-energy solutions provides guidance in constructing a more general basis system required to represent finite-energy solutions.

(5) Finally, the partially separated Bethe–Salpeter equation (1.1) is discretized by converting it into a generalized matrix eigenvalue equation for the coupling constant. One additional condition is imposed on the generalized matrix eigenvalue equation: In the zero-energy limit the generalized matrix eigenvalue equation that yields finite-energy solutions must devolve to the generalized matrix eigenvalue equation that yields zero-energy solutions. Discretization can be accomplished, for example, using the Rayleigh–Ritz–Galerkin method [21,22] or the method of orthogonal polynomials [14]. After expressing the solution $\Psi(\mathrm{i}p_0, p_s)$ in terms of basis functions, both sides of (1.1) are multiplied by $f(p_0, p_s)\, g_i(p_0, p_s)^{\dagger}$ and then integrated over the variables p_0 and p_s. The function $f(p_0, p_s)$ may be omitted or may be chosen so that that the matrices are symmetric or have some other desirable property. The integral equation (1.1) has then been converted into a generalized matrix eigenvalue equation

$$Kc = \frac{g_1 g_2}{4\pi}(V_{\mathrm{H}} + V_{\mathrm{AH}})c. \tag{1.3}$$

In the above equation, c is a column vector with the elements c_j that are the expansion coefficients for the wave function $\Psi(\mathrm{i}p_0, p_s)$ in (1.2), and the matrices V_{H} and V_{AH} are Hermitian and anti-Hermitian, respectively. Since the Bethe–Salpeter wave function is expressed in terms of N_{B} basis functions as indicated in (1.2), (1.3) is an $(N_{\mathrm{EQ}} \times N_{\mathrm{B}}) \times (N_{\mathrm{EQ}} \times N_{\mathrm{B}})$ matrix equation.

Because there is no obvious way to force the eigenvalues of (1.3) to be real, in general it has been extremely difficult to construct a generalized matrix eigenvalue equation that yields real values for $g_1 g_2/4\pi$ that are solutions to (1.1). A sufficient condition for obtaining real eigenvalues of a generalized matrix eigenvalue equation (1.3) is that $V_{\mathrm{AH}} = 0$, K be Hermitian and either K or V_{H} be positive definite. (See, for example, [23].) In (1.3), K is often Hermitian. But if K is also positive definite, then V_{AH} is usually non-zero, and if V_{AH} is zero, then neither K nor V_{H} is usually positive definite. And even if an eigenvalue of (1.3) happens to be real, especially when the basis functions do not obey the boundary conditions, the eigenvalue typically is not an eigenvalue of the Bethe–Salpeter equation (1.1).

Solutions to some partially separated Bethe–Salpeter equations have been obtained when the masses of the two constituents are equal because, in this case, the matrix K in (1.3) is both Hermitian and positive definite, and the matrix V_{AH} vanishes because it is proportional to the difference of the masses of the two

constituents. For example, for the equal-mass and zero-energy cases, the Bethe–Salpeter equation describing the Wick–Cutkosky model [5,6] can be converted into a matrix equation of the form (1.3) where both K and V_{H} are real, symmetric and positive-definite and $V_{\mathrm{AH}} = 0$ because it is proportional to both the mass difference of the two bound quanta and the energy of the bound state [24].

When a matrix eigenvalue equation is constructed such that the conditions discussed in steps (1)–(5) are satisfied, all eigenvalues usually are not real. But real eigenvalues are obtained, and almost all real eigenvalues are solutions of the original Bethe–Salpeter equation.

To demonstrate the techniques for solving a finite-energy, two-body, bound-state Bethe–Salpeter equation as well as the effectiveness of the method, finite-energy solutions are calculated for the partially separated Wick–Cutkosky model when the constituents masses are unequal. Although the equation is separable and the solutions were originally calculated from a completely separated equation, the method used here only requires that the two angular variables associated with rotations in three-dimensional space be separated. The advantage of demonstrating the technique with the Wick–Cutkosky model is that the complications associated with higher spin are avoided.

2. The Bethe–Salpeter equation for the Wick–Cutkosky model

The Wick–Cutkosky model [5,6] consists of two scalars with respective masses m_1 and m_2 that interact with a third massless scalar. In the ladder approximation, the Bethe–Salpeter equation that describes a bound state of the two massive scalars is

$$\begin{aligned}&\{(p^\mu + \xi K^\mu)(p_\mu + \xi K_\mu) - m_1^2\}\{[p^\nu + (\xi - 1)K^\nu][p_\nu + (\xi - 1)K_\nu] - m_2^2\}\chi_K(p)\\ &\quad = \frac{\mathrm{i}\lambda}{\pi^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4 q}{(p-q)^2 + \mathrm{i}\epsilon}\chi_K(q),\end{aligned} \tag{2.1}$$

where the notation is that of [25]. The parameter $0 < \xi < 1$ in the above equation is associated with the definition of the center-of-mass variables, and K^μ is the four momentum of the bound state. After a Wick rotation [5], in the rest frame of the center of mass where $K^\mu = (E, 0, 0, 0)$, the Bethe–Salpeter equation takes the form,

$$\begin{aligned}&\{(\mathrm{i}p^0 + \xi E)^2 - \mathbf{p}^2 - m_1^2\}\{[\mathrm{i}p_0 + (\xi - 1)E]^2 - \mathbf{p}^2 - m_2^2\}\chi_E(\mathrm{i}p_0, \mathbf{p})\\ &\quad = \frac{\lambda}{\pi^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4 \mathrm{q}}{(p-q)\cdot(p-q)}\chi_E(\mathrm{i}q_0, \mathbf{q}),\end{aligned} \tag{2.2}$$

where $(p-q)\cdot(p-q) \equiv (p_0 - q_0)^2 + (p^i - q^i)(p^i - q^i)$ is the Euclidean scalar product.

Dimensionless variables are introduced by defining $m_1 \equiv m(1 + \Delta)$, $m_2 \equiv m(1 - \Delta)$, dimensionless momentum $p' \equiv p/m$ and dimensionless energy $\epsilon \equiv E/2m$. When written in terms of dimensionless parameters, the above equation becomes

$$\begin{aligned}&\{(\mathrm{i}p^0 + 2\xi\epsilon)^2 - \mathbf{p}^2 - (1 + \Delta)^2\}\{[\mathrm{i}p_0 + 2(\xi - 1)\epsilon]^2 - \mathbf{p}^2 - (1 - \Delta)^2\}\chi_E(\mathrm{i}p_0, \mathbf{p})\\ &\quad = \frac{\lambda}{\pi^2 m^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4 \mathrm{q}}{(p-q)\cdot(p-q)}\chi_E(\mathrm{i}q_0, \mathbf{q}),\end{aligned} \tag{2.3}$$

where primes have been omitted since all momenta are now dimensionless.

For compactness of notation, it is convenient to write the coefficient of $\chi_E(\mathrm{i}p_0, \mathbf{p})$ on the left-hand side of (2.3) explicitly in terms of its real and imaginary parts,

$$\{(\mathrm{i}p^0 + 2\xi\epsilon)^2 - \mathbf{p}^2 - (1+\Delta)^2\}\{[\mathrm{i}p_0 + 2(\xi-1)\epsilon]^2 - \mathbf{p}^2 - (1-\Delta)^2\} \equiv D_R + \mathrm{i}D_I. \tag{2.4}$$

From (2.4) it immediately follows that

$$D_R = [p_0^2 + \mathbf{p}^2 - 4\xi^2\epsilon^2 + (1+\Delta)^2][p_0^2 + \mathbf{p}^2 - 4(1-\xi)^2\epsilon^2 + (1-\Delta)^2] + 16\xi(1-\xi)\epsilon^2 p_0^2, \tag{2.5a}$$

$$D_I = 4\epsilon p_0\{-\xi[p_0^2 + \mathbf{p}^2 - 4(1-\xi)^2\epsilon^2 + (1-\Delta)^2] + (1-\xi)[p_0^2 + \mathbf{p}^2 - 4\xi^2\epsilon^2 + (1+\Delta)^2]\}. \tag{2.5b}$$

Because D_I vanishes both in the zero-energy limit, $\epsilon = 0$, and, if $\xi = 1/2$, when the two constituents have equal masses, $\Delta = 0$, it is relatively easy to obtain solutions in these two limits.

Since the coupling constant λ is real in the Lagrangian, the physically interesting values of λ are real. Actually, for the Wick–Cutkosky model all eigenvalues are real [6,11,12] although for other Bethe–Salpeter equations, solutions may exist for complex values of the coupling constant as discussed previously [19,20].

Writing $\chi_E(\mathrm{i}p_0, \mathbf{p})$ in terms of real and imaginary parts,

$$\chi_E(\mathrm{i}p_0, \mathbf{p}) \equiv \chi_R(p_0, \mathbf{p}) + \mathrm{i}\chi_I(p_0, \mathbf{p}), \tag{2.6}$$

and noting that the real and imaginary parts of (2.3) must vanish independently, yields the following two equations:

$$D_R\chi_R(p_0, \mathbf{p}) - D_I\chi_I(p_0, \mathbf{p}) = \frac{\lambda}{\pi^2 m^2}\int_{-\infty}^{\infty} \frac{\mathrm{d}^4\mathrm{q}}{(p-q)\cdot(p-q)}\chi_R(q_0, \mathbf{q}), \tag{2.7a}$$

$$D_I\chi_R(p_0, \mathbf{p}) + D_R\chi_I(p_0, \mathbf{p}) = \frac{\lambda}{\pi^2 m^2}\int_{-\infty}^{\infty} \frac{\mathrm{d}^4\mathrm{q}}{(p-q)\cdot(p-q)}\chi_I(q_0, \mathbf{q}). \tag{2.7b}$$

Adding (2.7a) and (2.7b),

$$\begin{aligned} &D_R[\chi_R(p_0, \mathbf{p}) + \chi_I(p_0, \mathbf{p})] + D_I[\chi_R(p_0, \mathbf{p}) - \chi_I(p_0, \mathbf{p})] \\ &\quad = \frac{\lambda}{\pi^2 m^2}\int_{-\infty}^{\infty} \frac{\mathrm{d}^4\mathrm{q}}{(p-q)\cdot(p-q)}[\chi_R(q_0, \mathbf{q}) + \chi_I(q_0, \mathbf{q})]. \end{aligned} \tag{2.8}$$

From (2.3) it immediately follows that if $\chi_E(\mathrm{i}p_0, \mathbf{p})$ is a solution, then $\chi_E^*(-\mathrm{i}p_0, \mathbf{p})$ is a solution with the same eigenvalue. Thus, without loss of generality it is possible to choose

$$\chi_E(\mathrm{i}p_0, \mathbf{p}) = \chi_E^*(-\mathrm{i}p_0, \mathbf{p}). \tag{2.9}$$

Taking the complex conjugate of (2.9),

$$\chi_E^*(\mathrm{i}p_0, \mathbf{p}) = \chi_E(-\mathrm{i}p_0, \mathbf{p}). \tag{2.10}$$

Therefore, the real and imaginary parts of the solution can be chosen, respectively, to be even and odd functions of p_0.

Defining

$$\psi(p_0, \mathbf{p}) \equiv \chi_R(p_0, \mathbf{p}) + \chi_I(p_0, \mathbf{p}), \tag{2.11}$$

it immediately follows that

$$\psi(-p_0, \mathbf{p}) \equiv \chi_R(p_0, \mathbf{p}) - \chi_I(p_0, \mathbf{p}). \tag{2.12}$$

Consequently (2.8) can be rewritten as

$$D_R\psi(p_0,\mathbf{p}) + D_I\psi(-p_0,\mathbf{p}) = \frac{\lambda}{\pi^2 m^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}^4\mathrm{q}}{(p-q)\cdot(p-q)}\psi(q_0,\mathbf{q}), \tag{2.13}$$

which is in a form that is convenient to solve numerically.

3. Numerical solutions to the partially separated, Wick–Cutkosky model

Since (2.13) is manifestly invariant under rotations in three-dimensional space, the angular dependence associated with such rotations separates. The wave function $\psi(p_0,\mathbf{p})$ can be written in the form

$$\psi(p_0,\mathbf{p}) = F(p_0,|\mathbf{p}|)Y_m^\ell(\theta,\phi), \tag{3.1}$$

where $Y_m^\ell(\theta,\phi)$ is a spherical harmonic. The integration over the two angular variables on the right-hand side of (2.13) can be performed analytically using Hecke's theorem [26], and the angular dependence of the solution then separates. Unfortunately, with this approach the remaining kernel is an associated Legendre function containing a logarithmic singularity that is difficult to integrate over numerically [27]. Furthermore, the two remaining integrations on the right-hand side of (2.13) must be performed numerically.

An easier method for solving (2.13) is achieved by first rewriting the equation in terms of spherical coordinates in four-dimensional, Euclidean space–time [18,28]:

$$\begin{aligned} p^0 &= |p|\cos\theta_1, & p_z &= |p|\sin\theta_1\cos\theta_2,\\ p_x &= |p|\sin\theta_1\sin\theta_2\sin\phi, & p_y &= |p|\sin\theta_1\sin\theta_2\cos\phi. \end{aligned} \tag{3.2}$$

The four-momentum q^μ is written similarly in terms of primed angles.

The solution $\psi(p_0,\mathbf{p})$ is then expressed as a series expansion in terms of hyperspherical harmonics $P_{k,\ell}^{(2)}(\cos\theta_1)Y_m^\ell(\theta_2,\phi)$ in four-dimensional, Euclidean space–time. Defining $z \equiv \cos\theta_1$, the spherical function $P_{k,\ell}^{(2)}(z)$ is given by [29]

$$P_{k,\ell}^{(2)}(z) = (1-z^2)^{\ell/2}\frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell}C_k^1(z), \tag{3.3}$$

where $C_k^1(z)$ is a Gegenbauer polynomial. Now $C_k^1(z)$ is an even or odd function of z if the integer k is even or odd, respectively. From (3.3) it then follows that $P_{k,\ell}^{(2)}$ is an even or odd function of $\cos\theta_1$ if $k-\ell$ is respectively, an even or odd integer. Recalling that $\chi_R(p_0,\mathbf{p})$ and $\chi_I(p_0,\mathbf{p})$ are, respectively, even and odd functions of p_0, implying that they are also, respectively, even and odd functions of $\cos\theta_1$, zero-energy solutions can be obtained from expansions of the form

$$\chi_R(p_0,\mathbf{p})_{\text{zero-energy}} = \sum_{n=1}^{N_p} g_n G_n(|p|)P_{\ell+i,\ell}^{(2)}(\cos\theta_1)Y_m^\ell(\theta_2,\phi), \tag{3.4a}$$

$$\chi_I(p_0,\mathbf{p})_{\text{zero-energy}} = \sum_{n=1}^{N_p} g_n G_n(|p|)P_{\ell+1+i,\ell}^{(2)}(\cos\theta_1)Y_m^\ell(\theta_2,\phi). \tag{3.4b}$$

In the above expansions, g_n is an expansion coefficient, the index $i = 0, 2, \ldots$ is an even integer and $\{G_n(|p|)\}$ is a set of basis functions, each of which (very nearly) obeys the boundary conditions.

A generalization of the zero-energy basis system (3.4) that is suitable for calculating finite-energy solutions is

$$\chi_R(p_0,\mathbf{p}) = \sum_{n=1}^{N_p} \sum_{k=\ell,\ell+2,\ldots}^{K_{\max}} g_{n,k} G_n(|p|) P^{(2)}_{k,\ell}(\cos\theta_1) Y^\ell_m(\theta_2,\phi), \tag{3.5a}$$

$$\chi_I(p_0,\mathbf{p}) = \sum_{n=1}^{N_p} \sum_{k=\ell+1,\ell+3,\ldots}^{K_{\max}} g_{n,k} G_n(|p|) P^{(2)}_{k,\ell}(\cos\theta_1) Y^\ell_m(\theta_2,\phi). \tag{3.5b}$$

The values of the index k in (3.5) are chosen so that $\chi_R(p_0,\mathbf{p})$ and $\chi_I(p_0,\mathbf{p})$ are, respectively, even and odd functions of $\cos\theta_1$. In the above expansions, $g_{n,k}$ is an expansion coefficient and $\{G_n(|p|)\}$ is a set of basis functions, each of which (very nearly) obeys the boundary conditions and will be specified later. Recalling that $\psi(p_0,\mathbf{p}) = \chi_R(p_0,\mathbf{p}) + \chi_I(p_0,\mathbf{p})$ and using (3.5),

$$\psi(p_0,\mathbf{p}) = \sum_{n=1}^{N_p} \sum_{k=\ell,\ell+1,\ldots}^{K_{\max}} g_{n,k} G_n(|p|) P^{(2)}_{k,\ell}(\cos\theta_1) Y^\ell_m(\theta_2,\phi). \tag{3.6}$$

In the zero-energy limit, the angular dependence of the solution separates and only $\chi_R(p_0,\mathbf{p})$ or $\chi_I(p_0,\mathbf{p})$ is nonzero. Thus, zero-energy solutions can be obtained from (3.6) by choosing one value of the parameter $k = \ell, \ell+1, \ldots$, with each different value of k yielding different solutions. As a consequence, in the zero-energy limit the basis system (3.6) devolves to a suitable basis system for obtaining zero-energy solutions.

There are three advantages to seeking solutions of the form (3.6) instead of (3.1): (1) After using Hecke's theorem [26] to perform the three angular integrations analytically, the remaining kernel does not contain a logarithmic singularity. (2) In (2.13) only one integration must be performed numerically instead of two. (3) The basis functions have the correct angular dependence for zero-energy solutions so that fewer angular terms are required to obtain accurate, finite-energy solutions when the states are tightly bound.

Substituting (3.6) into (2.13),

$$\begin{aligned}
&\sum_{n=1}^{N_p} \sum_{k=\ell,\ell+1,\ldots}^{K_{\max}} g_{n,k} G_n(|p|)[D_R P^{(2)}_{k,\ell}(\cos\theta_1) + D_I P^{(2)}_{k,\ell}(-\cos\theta_1)] Y^\ell_m(\theta_2,\phi) \\
&\quad = \frac{\lambda}{\pi^2 m^2} \sum_{n=1}^{N_p} \sum_{k=\ell,\ell+1,\ldots}^{K_{\max}} g_{n,k} \int_0^\infty \mathrm{d}|q||q|^3 G_n(|q|) \times \int \frac{\mathrm{d}\Omega'_{(3)}}{(p^2+q^2-2pq\cos\Theta)} P^{(2)}_{k,\ell}(\cos\theta'_1) Y^\ell_m(\theta'_2,\phi'),
\end{aligned} \tag{3.7}$$

where Θ is the angle between the four-vectors p and q. Using Hecke's theorem [26] to perform the angular integration (all necessary formulas are in the appendix of [29]),

$$\begin{aligned}
&\sum_{n=1}^{N_p} \sum_{k=\ell,\ell+1,\ldots}^{K_{\max}} g_{n,k} G_n(|p|)[D_R P^{(2)}_{k,\ell}(\cos\theta_1) + D_I P^{(2)}_{k,\ell}(-\cos\theta_1)] Y^\ell_m(\theta_2,\phi) \\
&\quad = \frac{\lambda}{\pi^2 m^2} \sum_{n=1}^{N_p} \sum_{k=\ell,\ell+1,\ldots}^{K_{\max}} g_{n,k} \int_0^\infty \mathrm{d}|q||q|^3 G_n(|q|) \Lambda^{(2)}_k(|p|,|q|) P^{(2)}_{k,\ell}(\cos\theta_1) Y^\ell_m(\theta_2,\phi).
\end{aligned} \tag{3.8}$$

The function $\Lambda^{(2)}_k(|p|,|q|)$ is [26,29],

$$\Lambda^{(2)}_k(|p|,|q|) = \begin{cases} \frac{2\pi^2}{|p||q|(k+1)} \left(\frac{|q|}{|p|}\right)^{k+1} \text{if } |q| \leqslant |p|, \\ \frac{2\pi^2}{|p||q|(k+1)} \left(\frac{|p|}{|q|}\right)^{k+1} \text{if } |p| \leqslant |q|. \end{cases} \tag{3.9}$$

The dependence on the angular variables θ_2 and ϕ separates as it must.

To determine the boundary conditions, the parameters g_0 and g_∞ must be calculated that satisfy

$$G_n(|p|)\xrightarrow[|p|\to 0]{}|p|^{g_0}, \tag{3.10a}$$

$$G_n(|p|)\xrightarrow[|p|\to \infty]{}|p|^{-g_\infty}. \tag{3.10b}$$

Once the asymptotic behavior of integrals of the form

$$I(p)=\int_0^\infty \mathrm{d}|q||q|^n G(|q|)\Lambda_k^{(2)}(|p|,|q|), \tag{3.11}$$

which appears in (3.8), are determined, the boundary conditions are readily calculated. Specifically, the parameters i_0 and i_∞ must first be calculated that, respectively, satisfy

$$I(|p|)\xrightarrow[|p|\to 0]{}|p|^{i_0}, \tag{3.12a}$$

$$I(|p|)\xrightarrow[|p|\to \infty]{}|p|^{-i_\infty}. \tag{3.12b}$$

There are two possible values for the parameter i_0 in (3.12a) [15]:

$$\text{Solution IA}: \quad i_0=k, \quad -n+k+1\leqslant g_0, \quad n-k-1<g_\infty, \tag{3.13a}$$

$$\text{Solution IB}: \quad i_0=g_0+n-1, \quad -n-k-1<g_0<-n+k+1, \quad n-k-1<g_\infty. \tag{3.13b}$$

Similarly, there are two possible values for the parameter i_∞ in (3.12b) [15]:

$$\text{Solution IIA}: \quad i_\infty=k+2, \quad -n-k-1<g_0, \quad n+k+1\leqslant g_\infty, \tag{3.14a}$$

$$\text{Solution IIB}: \quad i_\infty=g_\infty-n+1, \quad -n-k-1<g_0, \quad n-k-1<g_\infty<n+k+1. \tag{3.14b}$$

Using the fact that as $|p|\to 0$, $D_R\to$ constant, $D_I\to|p|$ and substituting (3.10a) into (3.8), for Solution IA it follows that

$$|p|^{g_0}+|p||p|^{g_0}\sim|p|^k. \tag{3.15}$$

Thus, $g_0=k$. Because the smallest value of $k=\ell$,

$$G_n(|p|)\xrightarrow[|p|\to 0]{}|p|^{\ell}. \tag{3.16}$$

As can be readily checked, there are no other solutions for g_0. Similarly, at large $|p|$, the only solution is $g_\infty=k+6$ so

$$G_n(|p|)\xrightarrow[|p|\to \infty]{}\frac{1}{|p|^{\ell+6}}. \tag{3.17}$$

The knot structure is as follows: There are N_p cubic splines in the expansion (3.6) and N_p+4 momentum knots $T_p(i)$. To determine the momentum knots, N_p Chebyshev points $x_p(i)$ are calculated on the interval $-1<x_p(i)<1$,

$$x_p(i) = -\cos\frac{(2i-1)\,\pi}{2N_p}, \quad i = 1, 2, \ldots, N_p. \tag{3.18}$$

The momentum knots $T_p(i+4)$ are then given by

$$T_p(i+4) = C'\sqrt{\frac{1+x_p(i)}{1-x_p(i)}} + C'', \quad i = 1, 2, \ldots, N_p. \tag{3.19}$$

The constant C' is chosen by trial and error to approximately minimize the lowest zero-energy eigenvalue, and the constant C'' is chosen so that the first knot on the positive $|p|$-axis is not too close to $|p| = 0$. The values $C' = 1.0$ and $C'' = 0.01$ were satisfactory. A knot is placed at the origin, $T_p(4) = 0$, and the three knots $T_p(1)$, $T_p(2)$ and $T_p(3)$ are placed on the "negative" $|p|$ axis to allow maximum freedom in constructing the solution from splines near $|p| = 0$. The three knots on the "negative" $|p|$ axis are mirror images (about the origin) of the first three knots in (3.19). With this choice of knots, the first three splines are finite at the origin, creating sufficient freedom to construct solutions from splines near $|p| = 0$.

Angular knots are chosen on the z axis, where $z = \cos\theta_1$, so that numerical integrations can be carried out over $\cos\theta_1$. Defining $N_\theta \equiv K_{\max} - \ell + 1$, which is the number of hyperspherical harmonics in the expansion (3.6) of the solution, arbitrarily, but in analogy with splines, the number of angular knots T_z is chosen to be $N_\theta + 4$. The angular knots $T_z(1) = -1$, $T_z(N_\theta + 4) = 1$ and the remaining knots are the Chebyshev points

$$T_\theta(i+1) = -\cos\frac{(2i-1)\pi}{2(N_\theta+2)}, \quad i = 1, 2, \ldots, N_\theta + 2. \tag{3.20}$$

So that the basis functions $G_n(|p|)$ asymptotically vanish as indicated in Eqs. (3.16) and (3.17), $G_n(|p|)$ is chosen as follows:

$$G_n(|p|) = \frac{|p|^\ell}{a + |p|^{2\ell+5}} B_n(|p|) \equiv \mathscr{G}_\ell(|p|) B_n(|p|), \tag{3.21}$$

where $\mathscr{G}_\ell(|p|)$ is a convergence function, a is a constant and $B_n(|p|)$ is a cubic spline [30]. At small $|p|$, $G_n(|p|) \sim |p|^\ell B_n(|p|)$. Since the splines are also functions of $|p|$, at small $|p|$ the individual basis functions $G_n(|p|)$ do not exactly obey the boundary condition (3.16). However, as $|p| \to 0$, for each solution the sum of the first three splines in the expansion (3.6) approaches a constant so that each solution satisfies the boundary condition exactly. At large $|p|$, since all splines vanish, the convergence function is chosen to vanish as $1/|p|^{\ell+5}$, which is one power of $|p|$ slower than the rate in (3.17). But at large $|p|$, because the last spline does not decrease exactly as $1/|p|$, basis functions very nearly, but do not exactly, satisfy the boundary condition. Solutions can be obtained that obey the boundary condition exactly at large $|p|$ by extending momentum knots beyond $|p| = \infty$ [15] just as they were satisfied exactly by extending momentum knots below $|p| = 0$. Because solutions decrease so rapidly at large momenta, the value of solutions at very large $|p|$ has minimal impact on numerical solutions. As a consequence, for a given number of splines, more accurate solutions are obtained without using a knot structure that extends beyond $|p| = \infty$ and has fewer splines at small $|p|$ where the solution has most of its support.

To solve (3.8), the dependence on θ_2 and ϕ is first separated. Then the resulting equation is discretized using a hybrid method: The angular dependence is discretized using the method of orthogonal polynomials [14], which requires that the coefficient vanish independently for each of the first N_θ spherical functions $P^{(2)}_{\ell+I_{\theta-1},\ell}(z)$, $I_\theta = 1, \ldots, N_\theta$ in the equation. The product of functions that appear in the equation and spherical functions that appear in the expansion for solutions can be reexpressed as spherical functions, some of which have a larger first index. As a consequence, although there are N_θ different spherical

functions in the expansion for the solution, there are more than N_θ different spherical functions in the equation. Consequently, if a solution is to be obtained, the series must converge. Using the orthogonality relationship for the spherical functions $P^{(s)}_{i,j}$,

$$\int_{-1}^{1} \mathrm{d}z(1-z^2)^{\frac{s-1}{2}} P^{(s)}_{i,j}(z) P^{(s)}_{i',j}(z) = \frac{\pi\Gamma(i+j+s)}{2^{s-2}(2i+s)\Gamma(i-j+1)\Gamma^2(s/2)}\delta_{i,i'}, \tag{3.22}$$

it follows that multiplying the equation by $\sqrt{1-z^2}\, P^{(2)}_{\ell+I_{\theta-1},\ell}(z)$ and integrating over z achieves the desired discretization. The momentum dependence is discretized using a modified Rayleigh–Ritz–Galerkin method [21,22]. Thus (3.8) is converted into a generalized matrix eigenvalue equation of the form $Ag = \frac{\lambda}{m^2} Bg$, where the elements of the column vector g are the expansion coefficients $g_{n,k}$ in (3.6), by multiplying (3.8) by

$$|p|^{\mathscr{N}}\sqrt{1-z^2}\mathscr{G}_\ell(|p|)B_{I_p}(|p|)P^{(2)}_{\ell+I_{\theta-1},\ell}(z), \tag{3.23}$$

and integrating over z and $|p|$.

The expressions for the matrices $A_{i,j}$ and $B_{i,j}$ are, respectively,

$$\begin{aligned} A_{i,j} = {} & \int_{-1}^{1} \mathrm{d}z\sqrt{1-z^2}\int_0^\infty \mathrm{d}|p||p|^{\mathscr{N}}\mathscr{G}_\ell(|p|)B_{I_p}(|p|)P^{(2)}_{\ell+I_\theta-1,\ell}(z)\times[D_R P^{(2)}_{\ell+J_\theta-1,\ell}(z) \\ & + D_I P^{(2)}_{\ell+J_\theta-1,\ell}(-z)]B_{J_p}(|p|)\mathscr{G}_\ell(|p|), \end{aligned} \tag{3.24a}$$

and

$$\begin{aligned} B_{i,j} = {} & 2\int_{-1}^{1}\mathrm{d}z\sqrt{1-z^2}\int_0^\infty \mathrm{d}|p||p|^{(\mathscr{N}-1)}\int_0^\infty \mathrm{d}|q||q|^2\times\mathscr{G}_\ell(|p|)B_{I_p}(|p|)P^{(2)}_{\ell+I_\theta-1,\ell}(z) \\ & \times\frac{R(|p|,|q|)^{\ell+J_\theta}}{\ell+J_\theta}P^{(2)}_{\ell+J_\theta-1,\ell}(z)\,B_{J_p}(|q|)\,\mathscr{G}_\ell(|q|). \end{aligned} \tag{3.24b}$$

In (3.24b)

$$R(|p|,|q|) = \begin{cases} \frac{|q|}{|p|} & \text{if } |q|\leqslant|p|, \\ \frac{|p|}{|q|} & \text{if } |p|\leqslant|q|. \end{cases} \tag{3.25}$$

As compared with (3.6), indices have been changed in Eqs. (3.23) and (3.24) so that terms are automatically excluded when $i<j$ in $P^{(2)}_{i,j}$. Here $I_p = 1,\ldots,N_p$; $I_\theta = 1,\ldots,N_\theta$ and the index i is given by $i = N_p(I_\theta-1)+I_p$ with a corresponding expression for j. With the aid of the orthogonality relationship (3.22) for spherical functions, the integral over the variable z in (3.24b) can be performed analytically yielding

$$\begin{aligned} B_{i,j} = {} & \frac{\pi(2\ell+I_\theta)!}{(\ell+I_\theta)(I_\theta-1)!}\int_0^\infty \mathrm{d}|p||p|^{(\mathscr{N}-1)}\int_0^\infty \mathrm{d}|q||q|^2\times\mathscr{G}_\ell(|p|)B_{I_p}(|p|) \\ & \times\frac{R(|p|,|q|)^{\ell+I_\theta}}{\ell+I_\theta}B_{J_p}(|q|)\mathscr{G}_\ell(|q|)\delta_{I_\theta,J_\theta}. \end{aligned} \tag{3.26}$$

As can be seen from (3.26), if $\mathscr{N}=3$ the matrix B is both symmetric and positive definite. Also, the matrix A is symmetric when the quantity D_I vanishes, which, from (2.5b), occurs either when the energy is zero or when the masses of the constituents are equal. But when A and B are both symmetric and at least one is positive definite, all eigenvalues are real [23], so all eigenvalues are real for the two cases just mentioned. However, if the energy is finite and the masses are unequal, all eigenvalues of the discretized

equation are not real, but real eigenvalues are obtained. For the solutions of the discretized equation corresponding to the lowest six real eigenvalues, which were the only solutions checked, when a sufficient number of basis functions were used, the solutions to the discretized equation also satisfied the partially separated Bethe–Salpeter equation.

The disadvantage of choosing $\mathscr{N} = 3$ is that the generalized matrix eigenvalue equation actually represents the partially separated equation multiplied by $|p|^3$. The factor $|p|^3$ reduces the sensitivity of the matrix equation to the form of the solutions at small $|p|$ with the result that numerical solutions do not satisfy the partially separated equation as well in this region. Choosing $\mathscr{N} = 1$ allows accurate solutions to be calculated for small $|p|$. Because the same set of solutions is obtained when $\mathscr{N} = 1$ as when $\mathscr{N} = 3$, all solutions in Tables 1 and 2 are calculated with $\mathscr{N} = 1$.

The solutions obtained from the generalized matrix eigenvalue equation $Ag = \frac{\lambda}{m^2}Bg$ are checked in two ways: (1) As the number of basis functions $G_n(|p|)$ and $P^{(2)}_{i,j}(\cos\theta_1)$ are increased, the value of each eigenvalue must converge. (2) For each solution the left- and right-hand sides of (3.7) are compared in the physical region at the center of each rectangle in the grid formed by the angular knots and the momentum knots. By examining where the left- and right-hand sides of the equation agree least well, deficiencies are revealed and possible remedies can be efficiently tested. In addition, a reliability coefficient $r_{\text{lhs–rhs}}$ [31], which is a statistical measure of how closely the left- and right-hand sides agree at the $N_p \times (N_\theta + 3)$ points, is calculated. If the left- and right-hand sides agree exactly at every point, then $r_{\text{lhs–rhs}} = 1$.

Table 1 lists values of the coupling constant λ/m^2 that are calculated in the zero-energy limit ($\epsilon = 0$) when $m_1 = 4m_2$. Since the angular dependence separates in the zero-energy limit, only one angular basis function $P^{(2)}_{k,\ell}(\cos\theta_1)$ is used ($N_\theta = 1$). That single angular basis function is indicated by the value of the index $k = \ell$ in the sum (3.6). As the number N_p of momentum basis functions $G_n(|p|)$ is increased, the calculated values of the coupling constants converge to correct values, and the reliability coefficients $r_{\text{lhs–rhs}}$ approach unity. (The "exact" eigenvalues in Tables 1 and 2 are correct to at least four significant figures and are calculated numerically using a completely separated form of the Bethe–Salpeter equation [6].)

Table 2 lists values of the coupling constant λ/m^2 that are calculated for four values of the square of the normalized energy $\epsilon^2 \equiv [E/(m_1 + m_2)]^2 = 0.1,\ 0.5,\ 0.9$ and 0.99 when $m_1 = 4m_2$. For each energy, the number N_p of momentum basis functions and the number N_θ of angular basis functions used in the calculation are listed. As can be seen from Table 2, as the normalized energy ϵ increases from zero to unity (and the binding energy decreases to zero), even using additional basis functions it becomes increasingly difficult to obtain accurate eigenvalues. Nevertheless, when $\epsilon^2 = 0.99$, the first eigenvalue is readily calculated with a relative error of a few tenths of a percent, and the first six eigenvalues are all determined with relative errors less than five percent.

Table 1
Calculated values for the coupling constant λ/m^2 in the zero-energy limit when $m_1 = 4m_2$

$\lambda/m^2_{\text{exact}}$	ℓ	$N_p = 5$		$N_p = 10$		$N_p = 20$	
		$\lambda/m^2_{\text{calc}}$	$r_{\text{lhs–rhs}}$	$\lambda/m^2_{\text{calc}}$	$r_{\text{lhs–rhs}}$	$\lambda/m^2_{\text{calc}}$	$r_{\text{lhs–rhs}}$
1.838	0	1.905	0.9994	1.841	0.999990	1.838	0.99999968
5.000	0	5.775	0.9990	5.035	0.999993	5.000	0.99999972
5.654	1	5.753	0.9989	5.647	0.999996	5.652	0.99999995
9.817	0	11.53	0.9993	9.996	0.999993	9.822	0.99999957
10.43	1	11.71	0.9968	10.42	0.999991	10.42	0.99999989
11.46	2	11.51	0.9988	11.43	0.999986	11.45	0.99999982

Table 2
Calculated values for the coupling constant λ/m^2 when the energy is finite and $m_1 = 4m_2$

$\lambda/m^2_{\text{exact}}$	$\lambda/m^2_{\text{calc}}$	$r_{\text{lhs–rhs}}$
$\epsilon^2 = 0.1$, $N_p = 20$, $N_\theta = 10$		
1.686	1.686	0.99999973
4.690	4.691	0.99999975
5.156	5.154	0.99999987
9.252	9.264	0.99999954
9.688	9.669	0.99999960
10.42	10.41	0.9999973
$\epsilon^2 = 0.5$, $N_p = 20$, $N_\theta = 10$		
1.052	1.052	0.9999985
3.112	3.111	0.9999966
3.344	3.341	0.9999963
6.174	6.185	0.9999880
6.532	6.493	0.9999865
6.748	6.757	0.9999858
$\epsilon^2 = 0.9$, $N_p = 25$, $N_\theta = 20$		
0.3167	0.3165	0.99985
0.8500	0.8487	0.99973
1.550	1.547	0.99976
1.590	1.586	0.99942
2.534	2.522	0.99899
2.604	2.595	0.99918
$\epsilon^2 = 0.99$, $N_p = 30$, $N_\theta = 30$		
0.0702	0.0700	0.968
0.166	0.164	0.968
0.286	0.273	0.954
0.427	0.415	0.928
0.590	0.613	0.871
0.734	0.718	0.988

Earlier the author suggested an alternative method [2] for solving finite-energy, two-body, bound-state Bethe–Salpeter equations and then used the method to obtain solutions to the partially separated Wick–Cutkosky model. When $m_1 = 4m_2$ and $\epsilon^2 = 0.5$, the lowest five values of λ/m^2 calculated in [2] are 1.054, 3.067, 3.440, 5.834 and 6.792. The values obtained in [2] when $\epsilon^2 = 0.9$ are 0.3227, 0.9253, 2.161, and 3.077. Comparing eigenvalues calculated in [2] with those in Table 2, eigenvalues calculated here are more accurate, especially as λ/m^2 increases and as ϵ^2 approaches unity. The specialized basis functions required in [2] are difficult to calculate numerically and require significant computer time. Thus, the method discussed here yields more accurate results while utilizing a much less complicated numerical technique that runs more efficiently.

4. Conclusions

A systematic method is discussed for solving finite-energy, two-body, bound-state Bethe–Salpeter equations that does not require that the equation be completely separated or that the constituents have equal masses. To apply the method, an equation must first be Wick-rotated [5] and then the two angular variables associated with rotations in three-dimensional space must be separated, which is possible for many two-body, bound-state Bethe–Salpeter equations, including all such equations in the ladder

approximation. Zero-energy solutions are calculated first: The zero-energy equation is completely separated by expressing the solution as a product of a hyperspherical harmonic and a function $F(|p|)$ that depends only on the magnitude $|p| = (p_0^2 + \mathbf{p}^2)^{1/2}$ of the Euclidean four-momentum. The zero-energy solutions are then calculated by first expanding the function $F(|p|)$ in terms of basis functions that (very nearly) obey the boundary conditions and discretizing the equation by converting it into a generalized matrix eigenvalue equation that is solved numerically. It is important to calculate zero-energy solutions first because the basis functions that yield zero-energy solutions provide a guide for determining the basis functions that yield finite-energy solutions. Finite-energy solutions are calculated by expanding solutions in terms of basis functions, each of which is a product of a "convergence function" that typically obeys the boundary conditions, a hyperspherical harmonic in four-dimensional, Euclidean space and a spline that depends on the magnitude of the four-dimensional, Euclidean momentum. The basis functions that yield finite-energy solutions must devolve to the basis functions that yield zero-energy solutions in the zero-energy limit. The partially separated equation is then discretized and solved numerically by converting it into a generalized matrix eigenvalue equation. The generalized matrix eigenvalue equation that yields zero-energy solutions provides guidance in formulating a generalized matrix eigenvalue equation that yields finite-energy solutions, and the latter must devolve to the former in the zero-energy limit. Even though the coupling constants, which are calculated as eigenvalues of the generalized matrix eigenvalue equation, usually cannot all be forced to be real, real eigenvalues and corresponding solutions are obtained that satisfy the Bethe–Salpeter equation.

To demonstrate the techniques and utility of the method, when the constituents have unequal masses, finite- and zero-energy solutions are calculated to the partially separated Bethe–Salpeter equation describing the Wick–Cutkosky model [5,6]. For this particular equation it is convenient, but not essential, to discretize the angular dependence using the method of orthogonal polynomials [14] and the momentum dependence using a modified Rayleigh–Ritz–Galerkin method [21,22]. The advantage of demonstrating the techniques by solving the Wick–Cutkosky model is that complications associated with higher spin are avoided.

Using the numerical techniques presented in the paper, the author has begun obtaining finite-energy solutions to the scalar electrodynamics model [14,15,32] and to the scalar–scalar model [17–20] when the bound constituents have either equal or unequal masses. Thus, it is highly likely that the numerical method discussed here provides a means for obtaining general, finite-energy solutions to many two-body, bound-state Bethe–Salpeter equations.

References

[1] E.E. Salpeter, H.A. Bethe, A relativistic equation for bound-state problems, Phys. Rev. 84 (1951) 1232.
[2] G.B. Mainland, Numerical method for solving two-body, bound-state Bethe–Salpeter equations for unequal constituent masses, Few-Body Syst. 26 (1999) 27.
[3] R. Blankenbecler, R. Sugar, Linear integral equations for relativistic multichannel scattering, Phys. Rev. 142 (1966) 1051.
[4] E.E. Salpeter, Mass corrections to the fine structure of hydrogen-like atoms, Phys. Rev. 87 (1952) 328.
[5] G.C. Wick, Properties of Bethe–Salpeter wave functions, Phys. Rev. 96 (1954) 1124.
[6] R.E. Cutkosky, Solutions of a Bethe–Salpeter equation, Phys. Rev. 96 (1954) 1135.
[7] H.S. Green, Separability of a covariant wave equation, Nuovo Cim. 5 (1957) 866.
[8] K. Seto, On the normalization of the Bethe–Salpeter wave function in the unequal-mass case by means of the stereographic projection method, Prog. Theor. Phys. 39 (1968) 453.
[9] K. Seto, Erratum: On the normalization of the Bethe–Salpeter wave function in the unequal-mass case by means of the stereographic projection method, Prog. Theor. Phys. 40 (1968) 192.
[10] K. Seto, On the solution of the Bethe–Salpeter equation in the unequal-mass wick-cutkosky model in the moving system, Prog. Theor. Phys. 41 (1969) 1340.
[11] E. zur Linden, Solution of the Bethe–Salpeter equation for bound states – i, Nuovo Cim. 63A (1969) 181.

[12] E. zur Linden, Solution of the Bethe–Salpeter equation for bound states – ii, Nuovo Cim. 63A (1969) 193.
[13] B.J. Brennan, Unequal mass spinor-spinor Bethe–Salpeter equation, J. Math. Phys. 16 (1975) 2241.
[14] G.B. Mainland, Solving the two-body, bound-state Bethe–Salpeter equation, Computat. Phys. 192 (2003) 21.
[15] G.B. Mainland, The importance of boundary conditions in solving the bound-state Bethe–Salpeter equation, Few-Body Syst. 33 (2003) 71.
[16] J.L. Gammel, M.T. Menzel, Bethe-Salpeter equation: numerical experience with a hydrogen-like atom, Phys. Rev. A 7 (1973) 858.
[17] C. Schwartz, Solution of a Bethe–Salpeter equation, Phys. Rev. 137 (1965) B717.
[18] T. Nieuwenhuis, J.A. Tijon, O(4) expansion of the ladder Bethe–Salpeter equation, Few-Body Syst. 21 (1996) 167.
[19] W.B. Kaufmann, Numerical solutions of the Bethe–Salpeter equation, Phys. Rev. 187 (1969) 2051.
[20] N. Setô, I. Fukui, Scalar–scalar ladder model in the unequal-mass case. I. Numerical evidence for the complex eigenvalues, Prog. Theor. Phys. 89 (1993) 205.
[21] L.M. Delves, J. Walsh, Numerical Solution of Integral Equations, Clarendon Press, Oxford, 1974.
[22] K.E. Atkinson, A survey of numerical methods for the solution of Fredholm integral equations of the second kind, SIAM, Philadelphia, 1976.
[23] F.B. Hildebrand, Methods of Applied Mathematics, second ed., Prentice-Hall, Englewood Cliffs, NJ, 1965.
[24] G.B. Mainland, J.R. Spence, Numerical solutions to the partially separated, equal-mass, wick-cutkosky model, Few-Body Syst. 19 (1995) 109.
[25] J.D. Bjorken, S.D. Drell, Relativistic Quantum Fields, McGraw Hill, New York, 1965.
[26] E. Hecke, Über orthogonal-invariante integralgleichungen, Math. Ann. 78 (1918) 398.
[27] G.B. Mainland, Logarithmic singularities in two-body, bound-state integral equations, Computat. Phys. 174 (2001) 852.
[28] M. Leýy, Wave equations in momentum space, Proc. R. Soc. London Ser. A 204 (1950) 145.
[29] G.B. Mainland, Electromagnetic binding of a minimally interacting, relativistic spin-0 and spin-1/2 constituent: zero four-momentum solutions, J. Math. Phys. 27 (1986) 1344.
[30] C. de Boor, A Practical Guide to Splines, Springer-Verlag, Berlin, 1978.
[31] B.J. Winer, Statistical Principles in Experimental Design, McGraw Hill, New York, 1962.
[32] R. Sugano, Y. Munakata, On the Bethe–Salpeter equation for the scalar-spinor particle system, Prog. Theor. Phys. 16 (1956) 532.